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Abstract

One method to compute residues of zeta functions associated with prehomogeneous vector spaces is given with a typical example. It is based on the calculation of invariant hyperfunctions on prehomogeneous vector spaces.

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0 Introduction

We know that the calculation of functional equations of (global) zeta functions associated with prehomogeneous vector spaces is reduced to that of Fourier transforms of the complex powers of the relatively invariant polynomials. The next problem: how is the calculus of the residues of zeta functions ? We may easily see that the calculation of the Fourier transform of the "singular" invariant hyperfunction is important for the computation of the residues. It has been implicitly shown in Sato-Shintani [Sa-Sh] in the calculation of one example. However we face a lot of difficulty when we try to carry out the explicit calculation of the residues following their method. One difficulty is to handle the divergence on the process of the calculation and the other is to compute the Fourier transform of the singular invariant hyperfunctions. We have ,so far, no complete algorithmic method to control such divergence or to compute the Fourier transforms. We can find only some cases in which the calculation is possible by using the theory of holonomic systems and microlocal analysis. It is one of the important topics in the theory of invariant holonomic systems and hyperfunctions on prehomogeneous vector spaces.

In this paper, we give a brief explanation for this theory and give one example — "Shintani's zeta function". Shintani [Sh1] succeeded to evaluate part of the residues. We evaluate all of the residues in a different manner though some of them are conjectures. Of course, our result and Shintani's result coincides with each other in their intersection.

1 Review on Prehomogeneous Vector Spaces

1.1 Prehomogeneous Vector Spaces

Let $\boldsymbol{G}_{\mathbb{C}}$ be a complex reductive linear algebraic group ,$\boldsymbol{V}_{\mathbb{C}}$ a finite dimensional vector space and $\rho : \boldsymbol{G}_{\mathbb{C}} \longmapsto \boldsymbol{GL}(\boldsymbol{V}_{\mathbb{C}})$ a linear representation of $\boldsymbol{G}_{\mathbb{C}}$ to $\boldsymbol{V}_{\mathbb{C}}$.

Definition 1.1 (Prehomegeneous Vector Space) (1) *We say that* $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$ *is a* prehomogeneous vector space *if there exists a point* $x_0 \in \boldsymbol{V}_{\mathbb{C}}$ *such that* $\rho(\boldsymbol{G}_{\mathbb{C}}) \cdot x_0$ *is an* *open dense subset in* $\boldsymbol{V}_{\mathbb{C}}$

(2) *A polynomial* $f(x) \in \mathbb{C}[\boldsymbol{V}_{\mathbb{C}}]$ *is a relative invariant of* $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$ *if there exists a character* $\chi : \boldsymbol{G}_{\mathbb{C}} \longmapsto \mathbb{C}^{\times}$ *such that* $f(\rho(g) \cdot x) = \chi(g) f(x)$ *for all* $g \in \boldsymbol{G}_{\mathbb{C}}$ *We call it a* relative invariant *corresponding to the character* χ.

It is proved that any relative invariant of $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$ corresponding to the character χ is uniquely determined modulo a constant multiple.

We suppose the following conditions.

1. Any relative invariant of $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$ is an integer power of the irreducible relative invariant $P(x)$. We denote $n = \dim \boldsymbol{V}_{\mathbb{C}}$ and $d =$ degree of $P(x)$.

2. $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$ is regular ,i.e., $\det(\frac{\partial^2 P(x)}{\partial x_i \partial x_j}) \not\equiv 0$.

3. $\boldsymbol{V}_{\mathbb{C}}$ decomposes into a finite number of $\boldsymbol{G}_{\mathbb{C}}$-orbits.

Let $(\boldsymbol{G}_{\mathbb{C}}, \rho^*, \boldsymbol{V}^*_{\mathbb{C}})$ be the dual prehomogeneous vector space to $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$,i.e., $\boldsymbol{V}^*_{\mathbb{C}}$ is the dual vector space of $\boldsymbol{V}_{\mathbb{C}}$ and ρ^* is the contragredient representation of ρ. Then the triplet $(\boldsymbol{G}_{\mathbb{C}}, \rho^*, \boldsymbol{V}^*_{\mathbb{C}})$ also satisfies the above conditions. We denote by $Q(y)$ the irreducible relative invariant of $(\boldsymbol{G}_{\mathbb{C}}, \rho^*, \boldsymbol{V}^*_{\mathbb{C}})$. The degree of $Q(y)$ is same as that of $P(x)$. The corresponding character of $Q(y)$ is χ^{-1} ,i.e., $Q(\rho^*(g) \cdot x) = \chi^{-1}(g) Q(y)$.

We suppose one more assumption.

4. There exists an inner product $< x, y >$ on $x, y \in \boldsymbol{V}_{\mathbb{C}}$ such that $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$ and $(\boldsymbol{G}_{\mathbb{C}}, \rho^*, \boldsymbol{V}^*_{\mathbb{C}})$ have the same fundamental relative invariant,i.e., $P(x) = Q(y)$ by identifying $\boldsymbol{V}_{\mathbb{C}}$ and $\boldsymbol{V}^*_{\mathbb{C}}$.

Definition 1.2 (Real form) $(\boldsymbol{G}_{\mathbb{R}}, \rho, \boldsymbol{V}_{\mathbb{R}})$ *is a* real form *of* $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$ *if and only if the following conditions hold.*

1. $\boldsymbol{V}_{\mathbb{R}}$ *is a real form of* $\boldsymbol{V}_{\mathbb{C}}$.

2. $G_{\mathbb{R}} := G_{\mathbb{C}} \cap GL(V_{\mathbb{R}})$ *is a real form of* $G_{\mathbb{C}}$.

We denote by $G_{\mathbb{R}}^{+}$ the connected component of the real group $G_{\mathbb{R}}$.

1.2 Singular set and Singular orbit

The complement of the open orbit $\rho(G_{\mathbb{C}})\cdot x_0$ is denoted by $S_{\mathbb{C}}$. We call it the *singular set* of $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$. From the assumption 3, $S_{\mathbb{C}}$ decomposes into a finite number of orbits. Let

$$S_{1\mathbb{C}} \sqcup S_{2\mathbb{C}} \sqcup \ldots \sqcup S_{m\mathbb{C}} = S_{\mathbb{C}}$$

be the $G_{\mathbb{C}}$-orbital decomposition of $S_{\mathbb{C}}$. We call each $S_{i\mathbb{C}}$ a *singular orbit* of $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$. Let $G_{\mathbb{C}}^{1}$ be the subgroup of $G_{\mathbb{C}}$ defined by $G_{\mathbb{C}}^{1} := \{g \in G_{\mathbb{C}}; \chi(g) = 1\}$. We suppose that

5. $S_{\mathbb{C}} = \{x \in V_{\mathbb{C}}; P(x) = 0\}$ and each $S_{i\mathbb{C}}$ $(i = 1, \ldots, m)$ is a $G_{\mathbb{C}}^{1}$-orbit.

Let $S_{\mathbb{R}} := S_{\mathbb{C}} \cap V_{\mathbb{R}}$ and let $S_{\alpha\mathbb{R}} := S_{\alpha\mathbb{C}} \cap V_{\mathbb{R}}$ $(\alpha = 1, 2, \ldots, m)$. The real locus $S_{\alpha\mathbb{R}}$ decomposes into a finite number of connected components,

$$S_{\alpha\mathbb{R}} = \bigsqcup_{\beta=1}^{m_\alpha} S_{\alpha,\beta}.$$

Each connected component $S_{\alpha,\beta}$ is a $G_{\mathbb{R}}^{1}$-orbit where $G_{\mathbb{R}}^{1} := G_{\mathbb{C}}^{1} \cap G_{\mathbb{R}}^{+}$.

2 Local zeta functions and Their poles

2.1 Local Zeta Functions

Let $(G_{\mathbb{R}}, \rho, V_{\mathbb{R}})$ be a real form of $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ and let

$$V_1 \sqcup V_2 \sqcup \ldots \sqcup V_l = V_{\mathbb{R}} - S_{\mathbb{C}}$$

be the connected component decomposition of $V_{\mathbb{R}} - S_{\mathbb{R}}$. Each connected component V_i $(i = 1, 2, \ldots, l)$ is a $G_{\mathbb{R}}^{+}$-orbit. For a complex number $s \in \mathbb{C}$, consider the function on $V_{\mathbb{R}}$,

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & x \in V_i \\ 0 & x \notin V_i \end{cases} \tag{1}$$

for $i = 1, 2, \ldots, l$ with a complex parameter $s \in \mathbb{C}$. If the real part $\Re(s)$ is sufficiently large, $|P(x)|_i^s$ is a continuous function. It satisfies the equation

$$|P(\rho(g) \cdot x)|_i^s = |\chi(g)|^s |P(x)|_i^s$$

for all $g \in \boldsymbol{G}_{\mathbb{R}}^+$. Namely $|P(x)|_i^s$ is a relatively invariant function corresponding to the character $|\chi(g)|^s$.

We denote by $\mathcal{S}(\boldsymbol{V}_{\mathbb{R}})$ the space of rapidly decreasing functions on $\boldsymbol{V}_{\mathbb{R}}$. For $f(x) \in \mathcal{S}(\boldsymbol{V}_{\mathbb{R}})$, the integral

$$Z_i(f, s) = \int_{\boldsymbol{V}_{\mathbb{R}}} |P(x)|_i^s f(x) dx \qquad (i = 1, 2, \ldots, l)$$

is absolutely convergent if the real part $\Re(s) > -1$ and is a holomorphic function in $s \in \mathbb{C}$. It is continued to a meromorphic function on $s \in \mathbb{C}$. The map

$$f(x) \longmapsto Z_i(f, s) \qquad (f(x) \in \mathcal{S}(\boldsymbol{V}_{\mathbb{R}}))$$

defines a tempered distribution with a meromorphic parameter $s \in \mathbb{C}$. In fact, we see easily that $Z_i(Q^*(D_x)f, s+1) = b(s) Z_i(f, s)$ with a polynomial $b(s)$ called a *b-function.* This implies that $Z_i(f, s)$ is meromorphic in $\Re(s) > k - 1$ if it is meromorphic in $\Re(s) > k$. $Z_i(f, s)$ is a relatively invariant distribution ,i.e., $Z_i(f_g, s) = Z_i(f, s) |\chi(g)|^{-s-\frac{n}{d}}$ with $f_g(x) := f(\rho(g) \cdot x)$.

Theorem 2.1 (Sato-Shintani [Sa-Sh]) *The local zeta function $Z_i(f, s)$ has the following properties.*

1. *They have a functional equation of the form*

$$Z_i(f, s) = \sum_{j=1}^{l} c_{ij}(s) Z_j(f^\wedge, -s - \frac{n}{d}) \tag{2}$$

where $c_{ij}(s)$ are meromorphic functions in $s \in \mathbb{C}$ and $f^\wedge$ is the Fourier transform of f.

2. $Z_i(f,s)$ has possible poles in the set

$$\{s \in \mathbb{C}; b(s+k) = 0, k = 0,1,2,\ldots\}$$

The formula (2) is the Fourier transform of the relatively invariant distribution $|P(x)|_i^s$. The explicit computation of $c_{ij}(s)$ is often possible by analyzing the micro-local structure of $|P(x)|_i^s$. This formula (2) gives the functional equation of the global zeta function (see [Sa-Sh]).

2.2 Poles of Local Zeta Functions

The poles of $Z_i(f,s)$ are located in the set $\{s \in \mathbb{C}; b(s+k) = 0, k = 0,1,2,\ldots\}$. If $Z_i(f,s)$ has a pole at $s = \sigma$ of order k_σ, we ave the expression

$$Z_i(f,s) = \sum_{j=1}^{k_\sigma} (s-\sigma)^{-j} I_j^\sigma(f) + (\text{holomorphic part}).$$

The distribution $I_j^\sigma(f)$, appearing in the principal part of the Laurent expansion of $Z_i(f,s)$, are supported in the singular set $\boldsymbol{S}_{\mathbb{R}}$. Indeed, if f belongs to the space $C^\infty(\boldsymbol{V}_{\mathbb{R}} - \boldsymbol{S}_{\mathbb{R}})$ of compactly supported C^∞-functions on $\boldsymbol{V}_{\mathbb{R}} - \boldsymbol{S}_{\mathbb{R}}$, then $Z_i(f,s)$ is an entire function of $s \in \mathbb{C}$. It means that $I_j^\sigma(f) = 0$ for all $j = 1,2,\ldots,k_\sigma$. On the other hand, $Z_i(f_g,s) = Z_i(f,s)$ for all $g \in \boldsymbol{G}_{\mathbb{R}}^1$ with $\boldsymbol{G}_{\mathbb{R}}^1 := \boldsymbol{G}_{\mathbb{C}}^1 \cap \boldsymbol{G}_{\mathbb{R}}^+$. Then the distribution $Z_i(f,s)$ defines a $\boldsymbol{G}_{\mathbb{R}}^1$-invariant distribution. Namely, the distribution $I_j^\sigma(f)$ is supported in $\boldsymbol{S}_{\mathbb{C}}$ and invariant by the action of $g \in \boldsymbol{G}_{\mathbb{R}}^1$. From the result of [Mu1], we have the following fact: any $\boldsymbol{G}_{\mathbb{R}}^1$-invariant distribution supported in $\boldsymbol{S}_{\mathbb{R}}$ is given as a linear combination of $I_j^\sigma(f)$ if any relatively invariant distribution is written as a linear combination of $Z_i(f,s)$ $(i = 1,2,\ldots,l)$.

What we need in the computation of the residues of the global zeta functions is the Fourier transforms of $\boldsymbol{G}_{\mathbb{R}}^1$-invariant distributions whose supports are contained in $\boldsymbol{S}_{\mathbb{R}}$. Above all, the $\boldsymbol{G}_{\mathbb{R}}^1$ invariant measures on the $\boldsymbol{G}_{\mathbb{R}}^1$-orbits in $\boldsymbol{S}_{\mathbb{R}}$ are important. If they are written as a linear combination of $Z_i(f,s)$ $(i = 1,2,\ldots,l)$, then their Fourier transforms are computed from those of $|P(x)|_1^s, \ldots, |P(x)|_l^s$.

3 Global Zeta Functions and Their Residues

3.1 Zeta Integrals

Let $(\boldsymbol{G}_{\mathbb{R}}, \rho, \boldsymbol{V}_{\mathbb{R}})$ be a real form of the prehomogeneous vector space $(\boldsymbol{G}_{\mathbb{C}}, \rho, \boldsymbol{V}_{\mathbb{C}})$. In this section, we suppose that $\boldsymbol{G}_{\mathbb{C}}$ is a reductive group. We take a discrete subgroup $\boldsymbol{\Gamma}$ in $\boldsymbol{G}_{\mathbb{R}}^{+}$ and a lattice L in $\boldsymbol{V}_{\mathbb{R}}$ satisfying $\rho(\boldsymbol{\Gamma}) \cdot L \subset L$. For a function $f(x) \in \mathcal{S}(\boldsymbol{V}_{\mathbb{R}})$, we consider the integral

$$Z(f, s.L) := \int_{\boldsymbol{G}_{\mathbb{R}}^{+}/\boldsymbol{\Gamma}} \sum_{x \in L - \{P(x)=0\}} f(\rho(g) \cdot x) \chi(g)^{s} dg \tag{3}$$

where dg is the Haar measure on $\boldsymbol{G}_{\mathbb{R}}^{+}$. We suppose that the integral (3) is absolutely convergent for all $f \in \mathcal{S}$ if the real part $\Re(s)$ of s is sufficiently large. Then we have

$$Z(f, s.L) := \sum_{i=1}^{l} \xi_i(s, L) \int_{\boldsymbol{V}_{\mathbb{R}}} |P(x)|_i^{s-\frac{n}{d}} f(x) dx, \tag{4}$$

where

$$Z(f, s.L) := \sum_{[x] \in L \cap \boldsymbol{V}_i/\sim} \mu(x) \cdot |P(x)|^{-s}, \tag{5}$$

with $\mu(x) := \int_{\boldsymbol{G}_x^{+}/\boldsymbol{\Gamma}_x} d\nu_x$; $\sim$ stands for $\boldsymbol{\Gamma}$-equivalence; $\boldsymbol{G}_x^{+}$ and $\boldsymbol{\Gamma}_x$ stands for the isotropy subgroup at x of $\boldsymbol{G}^{+}$ and $\boldsymbol{\Gamma}$, respectively, and $d\nu_x$ is the invariant measure on $\boldsymbol{G}_x^{+}$.

The Dirichlet series $\xi_i(s, L)$ is absolutely convergent for $\Re(s) \gg 0$. Sato-Shintani's [Sa-Sh] main result is that $\xi_i(s, L)$ is extended as a meromorphic function in $s \in \mathbb{C}$ with a finite number of poles and has a functional equation. We call $\xi_i(s, L)$ is called a *zeta function* associated with the prehomogeneous vector space $(\boldsymbol{G}_{\mathbb{R}}, \rho, \boldsymbol{V}_{\mathbb{R}})$.

Now we shall try to evaluate the residues of $\xi_i(s, L)$. Suppose that $f(x) \in C_0^{\infty}(\boldsymbol{V}_{\mathbb{R}} - \boldsymbol{S}_{\mathbb{R}})$. Then we can divide the integral $Z(f, s, L)$ into two parts,

$$\begin{aligned} Z(f, s, L) &= \int_{\boldsymbol{G}_{\mathbb{R}}^{+}/\boldsymbol{\Gamma}} \sum_{x \in L} f(\rho(g) \cdot x) \chi(g)^{s} dg \\ &= \int_{\substack{\boldsymbol{G}_{\mathbb{R}}^{+}/\boldsymbol{\Gamma} \\ \chi(g) \geq 1}} \sum_{x \in L} f(\rho(g) \cdot x) \chi(g)^{s} dg \qquad (6) \\ &\quad + \int_{\substack{\boldsymbol{G}_{\mathbb{R}}^{+}/\boldsymbol{\Gamma} \\ \chi(g) \leq 1}} \sum_{x \in L} f(\rho(g) \cdot x) \chi(g)^{s} dg \qquad (7) \end{aligned}$$

We denote by $Z_+(f,s,L) := (6)$ and $Z_-(f,s,L) := (7)$. Then we see easily that $Z_+(f,s,L)$ is extended as an entire function in $s \in \mathbb{C}$. From the Poisson's summation formula, we have

$$Z_-(f,s,L) := \int_{\substack{\boldsymbol{G}^+_{\mathbb{R}}/\boldsymbol{\Gamma} \\ \chi(g)\le 1}} v(L)^{-1}\chi(g)^{s-\frac{n}{d}} \sum_{y\in L^*} f^\wedge(\rho^*(g)\cdot y)dg \tag{8}$$

where $x(L)$ is the volume of $\boldsymbol{V}_{\mathbb{R}}/L$, L^* is the dual lattice of L and $f^\wedge$ is the Fourier transform of f. We divide the integral (8):

$$\begin{aligned} Z_-(f,s,L) &= v(L)^{-1}\int_{\substack{\boldsymbol{G}^+_{\mathbb{R}}/\boldsymbol{\Gamma} \\ \chi(g)\le 1}} \chi(g)^{s-\frac{n}{d}} \sum_{y\in L^*-\boldsymbol{S}_{\mathbb{R}}} f^\wedge(\rho^*(g)\cdot y)dg & (9) \\ &\quad +v(L)^{-1}\int_{\substack{\boldsymbol{G}^+_{\mathbb{R}}/\boldsymbol{\Gamma} \\ \chi(g)\le 1}} \chi(g)^{s-\frac{n}{d}} \sum_{y\in L^*\cap\boldsymbol{S}_{\mathbb{R}}} f^\wedge(\rho^*(g)\cdot y)dg & (10) \end{aligned}$$

For the same reason of the entireness of $Z_+(f,s,L)$, the integral (9) can be extended as an entire function. The poles of $Z(f,s,L)$ are contained in the integral (10).

3.2 Arithmetic Part and Analytic Part

We can compute (10) a little more precisely under some suitable conditions. Note that the singular orbits are decomposed as

$$\boldsymbol{S}_{\mathbb{R}} = \bigsqcup_{\alpha=1}^{m}\bigsqcup_{\beta=1}^{m_\alpha} \boldsymbol{S}_{\alpha,\beta}.$$

We put

$$I_{\alpha,\beta}(f(\cdot)) := \int_{\boldsymbol{G}^1_{\mathbb{R}}} \sum_{y\in L^*\cap \boldsymbol{S}_{\alpha,\beta}} f^\wedge(\rho^*(g)\cdot y)dg.$$

Then

$$\begin{aligned} &\int_{\substack{\boldsymbol{G}^+_{\mathbb{R}}/\boldsymbol{\Gamma} \\ \chi(g)\le 1}} \chi(g)^{s-\frac{n}{d}} \sum_{y\in L^*\cap\boldsymbol{S}_{\mathbb{R}}} f^\wedge(\rho^*(g)\cdot y)dg \\ &= \int_0^1 t^{s-\frac{n}{d}} \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m_\alpha} I_{\alpha,\beta}(f(t\cdot))dt \\ &= \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m_\alpha} \int_0^1 t^{s-\frac{n}{d}} I_{\alpha,\beta}(f(t\cdot))dt,) \end{aligned} \tag{11}$$

if each of $I_{\alpha,\beta}(f(t\cdot))dt$ is an integrable function of t. We suppose that each $\boldsymbol{S}_{\alpha,\beta}$ admits a $\boldsymbol{G}^1_{\mathbb{R}}$-invariant measure $d\nu_{\alpha,\beta}$. Then $d\nu_{\alpha,\beta}$ relatively invariant measure on $\boldsymbol{S}_{\alpha,\beta}$:

$$d\nu_{\alpha,\beta}(\rho^*(g)\cdot y) = \chi(g)^{s_\alpha - \frac{n}{d}} d\nu_{\alpha,\beta}(y)$$

with some constant $s_\alpha \in \mathbb{C}$. Therefore,

$$I_{\alpha,\beta}(f(t\cdot)) = t^{s_\alpha} I_{\alpha,\beta}(f(\cdot)).$$

Then we have

$$\begin{aligned} (11) &= \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m_\alpha}\int_0^1 t^{s+s_\alpha-\frac{n}{d}} I_{\alpha,\beta}(f(t\cdot))dt \\ &= \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m_\alpha}\frac{1}{(s+s_\alpha-\frac{n}{d})} I_{\alpha,\beta}(f(\cdot))dt. \end{aligned}$$

Thus we have to evaluate $I_{\alpha,\beta}(f(\cdot))$ for the computation of the residues of $Z(f,s,L)$.

Moreover we can divide the integral $I_{\alpha,\beta}(f(\cdot))$ into the arithmetic part and the analytic part. That is to say, we have

$$I_{\alpha,\beta}(f(\cdot)) = \lambda_{\alpha,\beta}\int f^\wedge(y) d\nu_{\alpha,\beta}(y),$$

where

$$\lambda_{\alpha,\beta} = \sum_{[y]\in L^*\cap \boldsymbol{S}_{\mathbb{R}}/\sim} Vol(\boldsymbol{G}^1_y/\boldsymbol{\Gamma}_y). \tag{12}$$

Here $Vol(\boldsymbol{G}^1_y/\boldsymbol{\Gamma}_y)$ is the volume of the fundamental domain $\boldsymbol{G}^1_y/\boldsymbol{\Gamma}_y$ and $\sim$ means $\boldsymbol{\Gamma}$-equivalence. From the relatively invariance of the measure $d\nu_{\alpha,\beta}$, we have the formula of the Fourier transform

$$\int f^\wedge(y) d\nu_{\alpha,\beta}(y) = \int f(x)\sum_{i=0}^{l} c^i_{\alpha,\beta}|P(x)|_i^{s_\alpha-\frac{n}{d}}dx. \tag{13}$$

Thus we have the following formula.

Proposition 3.1 *The zeta function $\xi_i(s, L)$ has a simple pole at $s = -s_\alpha + \frac{n}{d}$ with the residue*

$$\sum_{\beta=1}^{m_\alpha} \lambda_{\alpha,\beta} \cdot c^i_{\alpha,\beta}$$

We call $\lambda_{\alpha,\beta}$ the arithmetic *part and $c^i_{\alpha,\beta}$ the* analytic *part.*

We are led to the following problem naturally.

Problem 1 *Evaluate the arithmetic part $\lambda_{\alpha,\beta}$ defined by* (12) *and the analytic part $c^i_{\alpha,\beta}$ defined by* (13)

In the next section, we give an example of the calculation of these parts.

4 An Example: Shintani's zeta function

4.1 Complex Prehomogeneous Vector Space

Let $G^{(n)}_{\mathbb{C}} := GL_n(\mathbb{C})$. and let $V^{(n)}_{\mathbb{C}} := Sym_n(\mathbb{R}) =$ the space of real symmetric $n \times n$ matrices.. The group action of $G^{(n)}_{\mathbb{C}}$ on $V^{(n)}_{\mathbb{C}}$ is given by

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t g$$

for $g = (g_{ij}) \in G^{(n)}_{\mathbb{C}}$ and $x = (x_{ij}) \in V^{(n)}_{\mathbb{C}}$. Then $(G^{(n)}_{\mathbb{C}}, \rho, V^{(n)}_{\mathbb{C}})$ is a prehomogeneous vector space with the singular set $S_{\mathbb{C}} := \{x \in V^{(n)}_{\mathbb{C}}; \det(x) = 0\}$. $P(x) := \det(x)$ is an irreducible relative invariant. The corresponding character is $\chi(g) = \det(g)^2$. We define the inner product $< x, y >:= Tr(x \cdot y)$ for $x, y \in V_{\mathbb{C}}$.

4.2 Real Form of Prehomogeneous Vector Space

We take the following real form.
$G^{(n)}_{\mathbb{R}} := GL_n(\mathbb{R})^+ = \{g \in GL_n(\mathbb{R}); \det(g) > 0\}$.
$V^{(n)}_{\mathbb{R}} := Sym_n(\mathbb{R}) =$ the space of real symmetric $n \times n$ matrices.
Then $(G^{(n)}_{\mathbb{R}}, \rho, V^{(n)}_{\mathbb{R}})$ is a real form of $(G^{(n)}_{\mathbb{C}}, \rho, V^{(n)}_{\mathbb{C}})$ with the singular set

$$S^{(n)}_{\mathbb{R}} := \{x \in V^{(n)}; \det(x) = 0\}.$$

We let

$$V_k^{(n)} := \{x \in Sym_n(\mathbb{R}); x \text{ has } k \text{ positive eigenvalues and } n-k \text{ negative eigenvalues}\}.$$

Then

$$\bigsqcup_{k=0}^{n} V_k^{(n)} := V_{\mathbb{R}}^{(n)} - S_{\mathbb{R}}^{(n)}$$

is the connected component decomposition.

We define the following measures.

$dx^{(n)} := |\bigwedge_{1\le i\le j\le n} dx_{ij}|$ for $x = (x_{ij}) \in V^{(n)}$, (the euclidean measure on $V^{(n)}$).

$dg^{(n)} := |\det(g)|^{-n}|\bigwedge_{1\le i\le j\le n} dg_{ij}|$ for $g = (g_{ij}) \in G^{(n)}$, (invariant measure on $G^{(n)}$).

$dg_1^{(n)}$:= an invariant measure on $SL_n(\mathbb{R})$ defined by $dg^{(n)} = dg_1^{(n)} \times \frac{d(\det(g))}{\det(g)}$.

4.3 Discrete groups and Lattices

We take the following discrete group and lattices.

$\Gamma^{(n)} := SL_n(\mathbb{Z})$

$L^{(n)} := \{x \in Sym_n(\mathbb{Z}/2);$the diagonals are elements of $\mathbb{Z}\}$

$L^{(n)*} := Sym_n(\mathbb{Z})$

Then $L^{(n)}$ and $L^{(n)*}$ are $\Gamma^{(n)}$-invariant sets.

$$L'^{(n)} := \begin{cases} L^{(n)} - (L^{(n)} \cap S) & \text{if } n \neq 2 \\ L^{(n)} - \{(L^{(n)} \cap S) \cup \{x \in L^{(n)}; \sqrt{-\det(x)} \in \mathbb{Q}\}\} & \text{if } n = 2 \end{cases}$$

$$L'^{(n)*} := \begin{cases} L^{(n)*} - (L^{(n)*} \cap S) & \text{if } n \neq 2 \\ L^{(n)*} - \{(L^{(n)*} \cap S) \cup \{x \in L^{(n)*}; \sqrt{-\det(x)} \in \mathbb{Q}\}\} & \text{if } n = 2 \end{cases}$$

Then they are also $\Gamma^{(n)}$-invariant sets.

4.4 Zeta Integrals

Let M be a $\Gamma^{(n)}$-invariant subset of $L'^{(n)}$ or $L'^{(n)*}$.

We set

$$Z(f, M, s) := \int_{G^{(n)}/\Gamma^{(n)}} (\det(g))^{2s} \sum_{x\in M} f(\rho(g)\cdot x)dg^{(n)},$$

for $f \in \mathcal{S}(\boldsymbol{V}^{(n)})$ where $\mathcal{S}(\boldsymbol{V}^{(n)})$ is the space of rapidly decreasing functions on $\boldsymbol{V}^{(n)}$. Then the integrals $Z(f, \boldsymbol{L}'^{(n)}, s)$ and $Z(f, \boldsymbol{L}'^{(n)*}, s)$ are absolutely integrable when the real part $\Re(s)$ of s is sufficiently large. We obtain the Dirichlet series $\xi_k^{(n)}(s, \boldsymbol{L})$ and $\xi_k^{(n)}(s, \boldsymbol{L}^*)$ by separating the Dirichlet series from the integrals.

$$Z(f, \boldsymbol{L}'^{(n)}, s) = \sum_{k=0}^{n} \xi_k^{(n)}(s, \boldsymbol{L}'^{(n)}) \int_{\boldsymbol{V}^{(n)}} |P(x)|^{s-((n+1)/2)} f(x) dx^{(n)}$$

$$Z(f, \boldsymbol{L}'^{(n)*}, s) = \sum_{k=0}^{n} \xi_k^{(n)}(s, \boldsymbol{L}'^{(n)*}) \int_{\boldsymbol{V}^{(n)}} |P(x)|^{s-((n+1)/2)} f(x) dx^{(n)}$$

These Dirichlet series $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)})$ and $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)*})$ are absolutely convergent for $\Re(s) > \frac{n+1}{2}$, and continued to the whole complex plane as meromorphic functions with poles at $s = 1, \frac{3}{2}, \ldots, \frac{n+1}{2}$. The order of these poles are 1 except for the case n=2. When n=2, the pole at $s = 1$ may not be simple.

On the other hand, we need the following Dirichlet series

$$\xi_{1b}^{(2)}(s, \boldsymbol{L}^{(2)}) := \sum_{\{x \in \boldsymbol{L}^{(2)}; \sqrt{-\det(x)} \in \mathbb{Q}\}/\Gamma^{(2)}} |\det(x)|^{-s}$$

$$\xi_{1b}^{(2)}(s, \boldsymbol{L}^{(2)*}) := \sum_{\{x \in \boldsymbol{L}^{(2)*}; \sqrt{-\det(x)} \in \mathbb{Q}\}/\Gamma^{(2)}} |\det(x)|^{-s}$$

for the evaluation of the residues of $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)})$ and $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)*})$. They are essentially the Riemann's zeta function.

4.5 Residues

Our problem is the following.

Problem 2 *For $n \geq 3$, evaluate the residues of $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)})$ and $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)*})$ in terms of the special values of $\xi_k^{(i)}(s, \boldsymbol{L}^{(i)}), \xi_k^{(i)}(s, \boldsymbol{L}^{(i)*})$ for $i \leq n-1$, and some other special values , for example , those of the gamma function $\Gamma(s)$, the volume of the fundamental domain $\boldsymbol{SL}_n(\mathbb{R})/\boldsymbol{SL}_n(\mathbb{Z})$ and so on.*

The residues of $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)})$ is given by the following formulas according our calculation. However some of them are now conjectures. See the Remark 4.1.

We denote by $Res_{s=s_i}(\xi_k^{(n)}(s, \boldsymbol{L}^{(n)}))$ the residue of $\xi_k^{(n)}(s, \boldsymbol{L}^{(n)})$ at $s = s_i$.

Case 1 ($i > 1$ **and** $i \neq n-2$)

$$Res_{s=\frac{i+1}{2}}(\xi_k^{(n)}(s, \boldsymbol{L}^{(n)})) =$$
$$2^{n(n-1)/2-1} \cdot (2\pi)^{-i(i+1)/4} \cdot Vol(\boldsymbol{SL}_i(\mathbb{R})/\boldsymbol{SL}_i(\mathbb{Z}))$$
$$\times \sum_{j=0}^{n-i} b_{ik}^{(n)j} \xi_j^{(n-i)}(\frac{n}{2}, \boldsymbol{L}^{(n-i)*})$$

Here ,Vol means the volume of the fundamental domain.

The values of $b_{ik}^{(n)j}$ *is given by the following formulas. We put*

$$b_{ik}^{(n)j} = 2^{-n(n-1)/4} \cdot (2\pi)^{(n+1)(2i-n)/4} \cdot \prod_{p=1}^{n-i} \Gamma(\frac{p}{2}) \cdot c_{ik}^{(n)j}$$

with

- *when* $n - i \equiv 0, j \equiv 0$ $(mod.2)$
  $$c_{ik}^{(n)j} = \exp(\tfrac{\pi}{4}\sqrt{-1}((n-i)(n-2k+2i)+2ij))\binom{\frac{(n-i)}{2}}{\frac{j}{2}}$$
- *when* $n - i \equiv 0, j \equiv 1$ $(mod.2)$
  $$c_{ik}^{(n)j} = 0$$
- *when* $n - i \equiv 1, j \equiv 1$ $(mod.2)$
  $$c_{ik}^{(n)j} = \exp(\tfrac{\pi}{4}\sqrt{-1}((n-i)(n-2k+2i)+2i(j+1)))\binom{\frac{(n-i-1)}{2}}{\frac{j-1}{2}}$$
- *when* $n - i \equiv 1, j \equiv 0$ $(mod.2)$
  $$c_{ik}^{(n)j} = \exp(\tfrac{\pi}{4}\sqrt{-1}((n-i)(n-2k+2i)+2i(j+1)+4k-2))\binom{\frac{(n-i-1)}{2}}{\frac{j}{2}}$$

Case 2 ($i = 1$ **and** $i \neq n-2$)

$$Res_{s=\frac{i+1}{2}}(\xi_k^{(n)}(s, \boldsymbol{L}^{(n)})) =$$
$$2^{-1} \cdot (2\pi)^{-1/2} \cdot \sum_{j=0}^{n-1} b_{1k}^{(n)j}(s)\xi_j^{(n-1)}(\frac{n}{2}+s, \boldsymbol{L}^{(n-1)*})|_{s=0}$$

The values of $b_{1k}^{(n)j}(s)$ *is given by the following formulas. We put*

$$b_{1k}^{(n)j}(s)=2^{-n(n-1)/4-(n-1)s}\cdot(2\pi)^{(-n^2+3n)/4}\cdot a_{1k}^{j}(s).$$

Then $a_{1k}^{j}(s)$ *is given by*

$$\sum_{j=0}^{n-1} a_{1k}^{j}(s)t^{j}=$$

$$(2\pi)^{-(n-1)/2}\prod_{p=1}^{n-2}\Gamma(s+\frac{p+1}{2})\exp(\frac{\pi}{4}\sqrt{-1}((n-1)n-2k))$$
$$(t^2\exp(-\pi\sqrt{-1}s)-\exp(\pi\sqrt{-1}s))^{\lfloor\frac{n-1}{2}\rfloor}\times(t\exp(-\frac{\pi}{2}\sqrt{-1}s)-\exp(\frac{\pi}{2}\sqrt{-1}\pi s))^{(1+(-1)^n)/2}$$

, *if* $k=n$ *and*

$$\sum_{j=0}^{n-1} a_{1k}^{j}(s)t^{j}=$$

$$(2\pi)^{-(n-1)/2}\prod_{p=1}^{n-2}\Gamma(s+\frac{p+1}{2})\exp(\frac{\pi}{4}\sqrt{-1}((n-1)n-2k))$$
$$\times(t^2\exp(-\pi\sqrt{-1}s)-\exp(\pi\sqrt{-1}s))^{\lfloor\frac{k}{2}\rfloor}\times(t^2\exp(\pi\sqrt{-1}s)-\exp(-\pi\sqrt{-1}s))^{\lfloor\frac{n-k-1}{2}\rfloor}$$
$$\times(t\exp(\frac{\pi}{2}\sqrt{-1}s)-\exp(-\frac{\pi}{2}\sqrt{-1}\pi s))^{(1+(-1)^{n-k})/2}$$
$$\times(t\exp(-\frac{\pi}{2}\sqrt{-1}s)-\exp(\frac{\pi}{2}\sqrt{-1}\pi s))^{(1+(-1)^{k+1})/2}$$

, *if* $k\neq n$. *Here* $\lfloor x\rfloor$ *stands for the greatest integer less than* x .

Case 3 ($i=n-2$, **namely the residues at** $s=\frac{n-1}{2}$)

$$Res_{s=\frac{n-1}{2}}(\xi_k^{(n)}(s,\boldsymbol{L}^{(n)}))=$$
$$2^{n(n-1)/4-1}\cdot(2\pi)^{(n-2)/4}\cdot\Gamma(\frac{1}{2})\cdot Vol(\boldsymbol{SL}_{n-2}(\mathbb{R})/\boldsymbol{SL}_{n-2}(\mathbb{Z}))\cdot\xi_{1b}^{(2)}(\frac{n}{2},\boldsymbol{L}^{(2)*})$$
$$+(**)$$

and

$$(**)=$$

$$2^{n(n-1)/2-1} \cdot (2\pi)^{-(n-1)(n-1)/4} \cdot Vol(\boldsymbol{SL}_{n-2}(\mathbb{R})/\boldsymbol{SL}_{n-2}(\mathbb{Z}))$$
$$\times \sum_{j=0}^{2} b_{n-2,k}^{(n)j} \xi_j^{(2)}(\frac{n}{2}, \boldsymbol{L}^{(2)*})$$

, if $n > 3$ *and*

$$(**) =$$
$$(2\pi)^{-1/2} \cdot \sum_{j=0}^{2} b_{1k}^{(n)j}(s) \xi_j^{(2)}(\frac{n}{2} + s, \boldsymbol{L}^{(2)*})|_{s=0}$$

, if $n = 3$ *. Here,* $b_{n-2,k}^{(n)j}$ *and* $b_{1k}^{(n)j}(s)$ *are the same ones given in the case 1 and the case 2 , respectively.*

Remark 4.1 *In the calculation of the residues, we must sometimes exchange the order of the limit and the integral. The author believes that they would be justified, but so far, the author has not the right proof in one place.*

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